APPENDICES

18.1. My journey.

I look upon mathematics not merely as a language, an art, and a science, but as a branch of philosophy, and regard the forms of reasoning which it embodies and enshrines as amongst the most valuable possessions of the human mind.
– James Sylvester
Mathematical works, vol. IV,
Lectures on the theory of reciprocants,
Lecture XXVII.

Two ways of looking at life, not necessarily mutually exclusive, are presented to us. The first is to treat it as an experience, and the second as a project.

Confronted with the finiteness of the human condition, what is to remain of our presence here? When self crumbles to dust, is our best achievement one of fame? But what is a name except a series of letters? I do not wish, out of choice, to leave a name for posterity, but ideas. Ideas can be outside of ourselves, they may correspond to the nature of the world, and it is possible to realise that we can be a detective in their discovery. If these remain, I shall have achieved something.

Someone once said in New Music Brighton, a composer's collective, that you do not have to be a great composer, all you have to be is yourself. So for the project which I have chosen, how can I organise my activities, whilst remaining true to the idea that in learning from the methods of those discoverers in science I hold in esteem, I still am and have to be myself?

In order to have a project, it is necessary to have goals. These are often subverted, but without a plan of action no random activity is likely to achieve significant objectives. I defined my objectives to become a thinker in theories of science and in philosophy at the age of 15. A career intervened, but our society does not recognise that people can be productive after the age of 50, so I have been presented with the fact that I can design my own activities, with enough income to do this, without interference from authority to work out my days in the commercial economy.

For these projects I have embraced, what comes naturally to me but not always to others is long-range planning and persistence to the point of obduracy in reaching goals. Many men and women are more intelligent than I am, but to complete a project intelligence is not always necessary. Anyway, I do not want to be thought of as superman, which to people who know me is a ridiculous idea; the existence of superman is always a lie, and people should watch out if it is ever claimed.

The project I have planned for the next three years, part of a project in mid-flow spanning the previous nine years, having completed an eBook *The climate and energy emergencies*, on The Assayer free eBook website also *Innovation in mathematics*, and this eBook, is to produce the mathematics eBook *Number, space and logic*, and with Graham Ennis the physics eBook *New physics*.

I have found that to complete research, quiet is useful, together with varied venues, and to have access to the literature when it is required.

In this process, the heresies I have discovered and promote trample rather severely on a large number of collective toes.

From the development of many ideas in this work, I have become aware that a feature of human thought is that collective understandings have taken precedence over independent analysis, and that once by historical processes an idea is embedded in the curriculum, it is very difficult to dislodge. Someone has remarked that my problems are not mathematical, they are political and social – organisations can claim authority for reasons of dogma. But my purpose is not to initiate a revolution, which is a consequence, rather I want to help the reader to understand the truth. If I am in error, so be it; the reader who can reason can determine the truth or falsity of what I say, however if errors have been made in collective understanding of what the subject is about, we are all human, but we must put things right.

After the research, my project is to promote the works of which this is one using social media, translation to other languages, teaching and a changeover to more practical activities.

This is my intention. I await to see how or if it is subverted.

I thought that my voyage had come to its end at the last limit of my power, – that the path before me was closed, that provisions were exhausted and the time come to take shelter in a silent obscurity.

But I find that thy will knows no end in me. And when old words die out on the tongue, new melodies break forth from the heart; and where the old tracks are lost, new country is revealed with its wonders.
– Rabindranath Tagore
Gitanjali, Song Offerings, 37.

18.2. Mathematical history.

Intricate numbers are more usually given the name split-quaternions or coquaternions. They were introduced by James Cockle in 1849 under the latter name in the London-Edinburgh-Dublin Philosophical Magazine.

Weber in *Lehrbuch der Algebra* [1895] employed the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ frequently, but appears not to identify it with the imaginary number i. In the 1920's Noether and E. Brauer [No83] gave a matrix representation of the quaternions, which contain the complex numbers as a subalgebra. The matrix representation of complex numbers is mentioned by Remmert in the popular book *Numbers* [Eb91].

Many writers employ 2×2 matrices not in the same representation as the intricate one, such as Frobenius [Fr73], Serre [Se03], Jacquet and Langlands [1JL70], [2La80] and Silverman and Tate [ST92]. Gelbart [Ge75], [Ge76] used effectively intricate numbers in computations in the 1970's.

I introduced the term intricate number in 2008 to describe associative division algebras and extensions of Galois theory to matrix variables in a form extending the complex numbers, and therefore with a similar name. The hyperintricate representation may have been initiated in 2008.

These were used to prove for associative divisions algebras, which are described by matrices, that up to isomorphism the most general are the quaternions, of dimension 4. The existence of Wedderburn's little theorem, that every finite division ring is commutative, considerably surprised me, so that I decided to prove it by my own methods. It was only later that I looked at combining these two ideas together to see whether it was possible to prove the Feit Thompson theorem, that every nonabelian simple group is of even order.

It has interested me for many decades that the number of nonassociative division algebras is limited to the octonions up to isomorphism. The 10-novanion algebra came to me in seeking a proof of this fact. Initially I thought I had a counterexample, and this generated some interest, but Doly García came up with her own counterexample to what I was saying, and I realised that a calculation I had done had improperly skipped the details for the 'a = 0' case, but was otherwise OK. This has fascinated me further, since it is possible to describe physics as a state at a time t = 0 which does not conserve number, but for which at all other scalar time variables, number is conserved. These are the features of a novanion algebra. The idea that the novanion algebra could be used in physics arose from a conversation with Daniel Hajas, a first year physics student at the University of Sussex, who suggested the n = 10 case of the novanions could be linked with the 10 dimensional heterotic string. This led me to look for an n = 26 dimensional case for bosonic strings, and this was found, but not in the way I originally expected.

The discussion on ladder numbers is a development of the eBook *Innovation in mathematics*. The theory was developed in response to questions by Tim Gibbs, and I decided to articulate disquiet that I felt about the consistency of the real number system. The resulting theories met with considerable resistance, since one of them violates current understandings. I have tried to explain in great detail an exposition of these ideas. It is true that $\mathbb{N}^{\mathbb{N}} \equiv \mathbb{N}$ and representable infinitesimals are not accepted by Doly García, and that I have kept my stance in what I have to present. On the first point, the proof does not fall down because of indistinguishable copies of $\mathbb{N}$, and on the second, arithmetic has the same consistency as arithmetic with representable infinitesimals. If what one is saying is not accepted, I think it is necessary to keep integrity and state a sound belief that all objections have been dealt with satisfactorily. A positive consequence of this critique is that I have had to respond to the demand of proving the consistency of all statements I make. This issue has been addressed in chapter XIV, with the result that the frontiers of my own knowledge and those of the mathematical community have been pushed back, with interesting new conclusions.

Discussing the quintic equation and its unsolvability by Galois theory, G.B. Mathews in [1Ma30] has said that Jerrard was the last disputant. No blame should be apportioned to 19th century mathematicians in the adoption of Galois theory, which had met with few theoretical or practical obstructions. Wedderburn's proof of his little theorem in 1905 that finite division rings are commutative should have indicated that solvability in the case of rings needed further investigation. Up to the present day this has remained a muted concern, but a question arises whether conformity to an embedded feature of the curriculum has inhibited free enquiry. That the solvability question could be resolved in part using the theory of varieties developed amongst others by Hilbert seems not to have been voiced, possibly because the theory of groups had displaced other explanations. Amongst contemporary mathematicians, I am aware only of J. S. Milne who notes in a more general setting a mismatch between the groups of Galois theory and those of varieties. Looking back at my work on the subject, leading in chapters VIII to X to the deconstruction of Galois theory, its reformulation as a theory of varieties and the attempt at the sextic by comparison methods in chapter XI, I find it difficult to understand why I have been so obsessed with a project on deconstructing a theory which has been almost universally accepted. This is not rational activity, it is intuitive.

Personally, my deconstruction of Galois solvability theory has met obstacles, presented in the following dialogue derived from the content of several conversations.
R.G.: Have you read Artin's book on Galois theory?
Jim: Three or four times. I have read about forty books on Galois theory, and I do not understand any of them.
R.G.: That is about the maximum number of books on the subject. von Neumann claimed: "in mathematics you don't *understand* things. You just *get used* to them". Do the exercises!
Jim: Here, as sometimes elsewhere, I do not accept what von Neumann is saying. As you know, in general I do not do unnecessary drill. There is a substitute here, however. I have made between 5,000 and 10,000 pages of computations on the subject. My investigations show that the theory of inseparable extensions is a subcase of the theory of dependent roots I have developed, that, as is well known, there are matrix solutions of complex polynomials of arbitrary degree, which directly contradicts Galois theory, and that ring automorphisms, which is the correct way of describing complex transformations of polynomial equations in additive format, the fact being that in multiplicative format the polynomial is already solved, these automorphisms are involutions in pairs of roots which do not leave other roots intact, and so cannot be described by permutations, as is done in Galois theory. An end result of Galois theory, that the general quintic and higher degree polynomials with complex solutions are unsolvable by radicals, is found independently of Galois theory by looking at the theory of varieties in a special case – killing central terms, and by an extension of this, but the Galois group mapping to the theory of varieties cannot be saved. The disconnection of this mapping means that large parts of mathematics, first developed by Ruffini, Galois, Jordan, Hölder, Schreier and E. Artin, are wrong.
R.G.: Anyone making such claims is insane. Substantial portions of modern mathematics are based on Galois theory, and practically every mathematician accepts it.
Jim: No. I am not disputing Galois representation theory, which is a distinct topic. It is a general process of history that there are people who are comfortable in accepting the system, and others in the pursuit of understandings which do not conform who suffer from it. A general assumption is that we are prepared to admit that ideas of classical antiquity may be in error, but we are not prepared to apply the same attributions to our own age. But the thesis I wish to promote is that in three areas, the consistency of the real number system, decidability and Galois theory, the mathematics of today is erroneous. These errors have persisted for well over eighty years. No such analogous situation arose when the highest development in geometry was the geometry of Euclid. We need to ask ourselves why.

The idea of $\mathbb{D}$w exponential algebras originates from my own work in the 1980's, or rather before, unfortunately in the middle of an examination. A possible application to the Riemann hypothesis was communicated to me by the physicist David Bohm via a third party, Ebrahim Baravi, a physics student at Sussex University with whom I had conversations on quantum gravity, quantum thermodynamics and whether entropy was increasing or not in gravitational interactions. David invited me to become his research assistant, but I thought that the fact that

I had no University degree introduced complications, and I turned him down. David died of cardiac arrest before an investigation of the Riemann conjecture could be properly developed. I continued my work outside the academic system, not returning to the problem for over thirty years. When w is a natural number these methods have turned out to be ineffective for its solution, also for analytic continuation in ladder algebra, so we have looked at other ways of extending standard theory.

18.3. The teaching of mathematics. [2We79].

The essence of creative mathematics is not just to replicate, but to reconstruct, deconstruct and extend.

Students are sometimes marked on the production of results that are claimed valid by those asking the questions. I believe this is the wrong approach. The student needs to articulate his or her insight. When asked a question the correct approach is to look at the data, irrespective of the result which entails. To solve a problem correctly may take far longer than the system allows. The collective wisdom may be erroneous, and what is needed is the development of persistent curiosity.

Edison claimed that invention is one percent inspiration and ninety nine percent perspiration; Einstein claimed that insanity is doing the same thing over and over, getting the same result. Results are rarely obtained on a first try, and the researcher needs to ask whether there are resources to complete the task at the hundredth attempt. The student must become aware that there are social barriers to what can and cannot be investigated, that it takes courage to ignore them, that results cannot usually be obtained without persistence, nor great results without persistence verging on insanity.

The reader may judge how far I have reached my own standards. Examples should be provided to give meaning to theorems and the trajectory of the development of a proof should be discussed before it is initiated. Knowledge of why a proof is in the form it is presented is as important as how the proof is constructed. The context and significance of a proof should be addressed. ‘Transparent brain syndrome’, where the writer assumes the reader knows what the writer is thinking about without explaining it, should be weeded out.

Although I think examples should come before generalities, that it is better to build up in stages rather than present a full-blown theory all at once, and that an understanding of the history of a mathematical idea is often useful, André Weil makes some good points in this viewpoint to the Indian Mathematical Society in 1931.

(1) The study of mathematics, as well as of any other science, consists of the acquisition of useful reflexes and independent habits of thought. The acquisition of useful reflexes should never be separated from the perception of their usefulness.

It follows that problem-solving should never be practiced for its own sake, and particularly tricky problems must be excluded altogether. The purpose of problems is twofold: either to drill the student in the application of some method of special importance, or to develop his originality by guiding him along some new path. Drill is essentially a school-method, and ought to become unnecessary in the final stages of University teaching.

(2) Rigour is to the mathematician what morality is to man. It does not consist in proving everything, but in maintaining a sharp distinction between what is assumed and what is proved, and in endeavouring to assume as little as possible at every stage.

The student should therefore be gradually accustomed, by means of startling examples, to question the truth of every unproved proposition, until at least he is able to deduce from the ordinary axioms everything he has learnt.

(3) Knowledge of a proof means the understanding of its machinery and the ability to reconstruct it. This implies
(i) perfect correctness in the definitions,
(ii) a faculty of connecting with the general ideas underlying it,
(iii) a perception of the logical nature of any proof.

The teacher should therefore follow, not the quickest nor even the most elegant method, but the method which is related to the most general principles. He should also point out everywhere the relation between the various elements of the hypothesis and conclusion. Students must be accustomed to draw a sharp distinction between premises and conclusion, between necessary and sufficient conditions, and between a theorem and its converse.

(4) The teaching of mathematics must be a source of intellectual excitement. This can be achieved, at the higher stages, by taking the student to the brink of the unknown, and at earlier stages by making him solve for himself questions of theoretical or practical importance.

This is the method followed in the “seminars” of the German Universities, first organised by Jacobi a century ago, and even now the most prominent feature of the German system. Division of labour between students in the study of a given group of questions is a common practice in these seminars, and proves to be a powerful incentive to work.

18.4. Communicating mathematical research.

Mathematics, as well as being a social system, after selection of an axiom system and its rules of deduction, concerns itself with universal truths whose validity is beyond human direction or control.

Stated simply, it is not possible to be human and not make mistakes. We have to admit that the founders of our subject, on whom all subsequent work has been based, are prone to embarrassing error, and sometimes not infrequently. James Sylvester once admitted that his mathematical work was occasionally in error, and pointed out that the choice was between allowing others to correct his work or submitting severely curtailed output. The mathematical superstar Henri Poincaré had to submit five supplements to correct his work which became the foundations of homology theory.

Mathematics of the present day can pursue two strategies in the face of the complexity of the results of investigation. One is the abstract approach, where lives the hope that problems may be solved without, on the face of it, employing much computation. This is a stream started most notably by Hilbert, to whom the previous paragraph applies, and its manifestations continue today. The second is the use of brute force modulated by human intelligence. By the means of computer software previous areas of investigation which had to apply the abstract approach to maintain progress, became amenable to massive calculation. For instance, it is probably only for reasons of history that the incidence matrix approach to homology theory, now mainly forgotten, is not reinstated as a computationally easy and direct method.

It may be felt that the peer review system now adequately protects mathematics from the mistakes of its forebears. Ironically Riemann's work 'On the number of primes less than a given magnitude' would nowadays not have passed the peer review process in the form which it is presented to us.

Mathematicians, like no other, are as a whole strongly aware of the necessity of avoiding error, and many would say of absolutely avoiding it. Yet there is much work in the current era, whilst bypassing (not always) the pitfalls of linguistic sociology or literary criticism, that falls short of what is desirable.

The system itself is at fault, not only in the rapacious actions of the publisher Reed Elsevier, but also because the peer review process will often resist publication of a result unless its text intimidates the referee. A system which rejects work that is too simple, and accepts the same results when the paper (usually salami-sliced) is rewritten in more technical and elevated language is not acting correctly. Further, some rigid criterions allow the use of other work that is peer reviewed. It is then possible for an article to use a large number of peer reviewed sources, the probability in each of which there remain errors is not zero, and just as frequently a writer will misstate the conditions under which quoted theorems are said to hold.

The peer review service may stumble in other ways. An overly abstract and intimidating style to a mathematical subject should raise suspicion, but is not of itself a proof of insecure foundations. Some 'abstract mathematical nonsense' is developed within a number of leading institutions. Moreover, there seems to be a whole industry converting classical results from the nineteenth century into homological and homotopical form where their origin and significance are obscured. As a separate issue, I would like to indicate my opinion of the work of Nathan Jacobson (1910 – 1999, who taught at Yale from 1947 and was president of the AMS 1971 – 1973). It follows from the work of Gentzen that all proofs may be put in the form of a tree, where the top of the tree contains the assumptions of the proof, and the root contains the conclusion. So far as I can gather, all proofs by Jacobson contain internal loops, that is, they cannot be reduced to tree form. It may happen that a proof branches off into other proofs which themselves contain circuits, or nodes which are ambiguous or absent. Thus it appears that whereas Jacobson's work contains many true theorems (and a few false ones), the proofs are invalid.

The system can inhibit work that is critical of its output. Cumulatively, and despite the wide availability of many accessible sources of information, the result is that the technical mathematical literature is blocked to an audience that does not have access to purchase its information or the means to see through its elevated language to the true content of which it speaks. As a consequence mathematics and other sciences have become alienated from the general population, and except for the work of a number of academics who explain technicalities rather than hide behind them, its modern and detailed truths are sometimes unknown and unavailable to the citizen outside a closed system. We can raise the question as to whether mathematics in its establishment form, as a discipline of knowledge, truth and unfettered investigation, as a percentage of the ever increasing number of mathematicians contributing to the subject, is in these ways in decline.

ANSWERS TO EXERCISES

I have provided answers to questions where I know them, but I have not investigated all aspects of the subject. So where there are no answers given, you, the reader, are invited to become a researcher and a creator of your own theories.

Chapter I.

(A) *Check*

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix} = \begin{bmatrix} 0 & ac \\ bd & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & cb \\ ad & 0 \end{bmatrix}.$$

(B) *Find a way to memorise the intricate basis element equations* (1) *of section* 1.6.

If we order the intricate basis as

$1, i, \alpha, \phi,$

then $1 \times$ anything commutes. For i, α and ϕ, any product of two of these intricate numbers anticommutes, for example

$i\alpha = -\alpha i.$

To look at the different products of i, α and ϕ, taken in that order, the only pair which gives a minus sign is $i\alpha = -\phi$.

What are the values of i^2, α^2 and ϕ^2?

(C) *Define triticate numbers by the following basis elements* (ω *is a third root of unity* = $e^{2\pi i/3}$ = $\cos(2\pi/3) + i\sin(2\pi/3)$, *where* $\omega^3 = 1$).

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \omega^2 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega^2 \\ \omega & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ \omega & 0 & 0 \\ 0 & \omega^2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ \omega^2 & 0 & 0 \\ 0 & \omega & 0 \end{bmatrix}.$$

Show this basis is linearly independent.

The numbers 1, ω, and ω^2 are not linearly independent, because, using the representation from trigonometry

$\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i,$

(how can you prove this?), we have

$1 + \omega + \omega^2 = 0.$

The definition of linear independence is given in chapter I, section 3, where for linear dependence in the example there have to be three numbers a, b and c, not all zero so that

$a1 + b\omega + c\omega^2 = 0.$

Nevertheless, for intricate numbers, although 1 and -1 are linearly dependent, the basis

$$1 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \text{ and } \alpha = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

is linearly independent, and -1 is a square root of unity. Thus a linearly independent basis can be built out of the numbers

$$1\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \text{ and } \alpha\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}.$$

We can carry this over to the basis in

$$\begin{bmatrix}1 & 0 & 0\\0 & \omega & 0\\0 & 0 & \omega^2\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\0 & (\omega)^2 & 0\\0 & 0 & (\omega^2)^2\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\0 & (\omega)^3 & 0\\0 & 0 & (\omega^2)^3\end{bmatrix}.$$

Multiplying the first matrix by a, the second by b and the third by c, and setting the sum to zero, gives three linear equations which cannot be satisfied unless a, b and c = 0.

(D) *Are there any other representations like the above using cube roots of unity that you can construct*?

(E) *Develop the theory of triticate numbers by analogy or otherwise with the treatment of intricate numbers*.

(F) *Are there other representations, say for penticate numbers for prime* p = 5, *other prime numbers, and composite numbers* (*a product of primes*)?

(D) – (F) The first row in the definition of triticate numbers is of the form $\begin{bmatrix}x & 0 & 0\\0 & y & 0\\0 & 0 & z\end{bmatrix}$. Choose the first item, this is $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$. Under matrix multiplication this represents an identity permutation under composition of permutations, $\begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix}$.

The matrix $\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\1 & 0 & 0\end{bmatrix}$ represents the permutation $\begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix}$, and the matrix $\begin{bmatrix}0 & 0 & 1\\1 & 0 & 0\\0 & 1 & 0\end{bmatrix}$ the permutation $\begin{pmatrix}1 & 2 & 3\\3 & 1 & 2\end{pmatrix}$. On the same ordered set of elements, these *cyclic* permutations are *abelian* (look up definitions of cyclic and abelian if you do not understand these). But this is *not* all permutations of a 3 × 3 matrix. For instance $\begin{pmatrix}1 & 2 & 3\\1 & 3 & 2\end{pmatrix}$ is a permutation, but it is not cyclic on three elements (it is cyclic on two elements).

We now introduce two ideas. The first is to convert matrices from the diagonal related form, say $\begin{bmatrix}x & 0 & 0\\0 & y & 0\\0 & 0 & z\end{bmatrix}$ to antidiagonal form $\begin{bmatrix}0 & 0 & x\\0 & y & 0\\z & 0 & 0\end{bmatrix}$. Clearly this mapping, U, is an involution (because $U^2 = 1$ we obtain the original matrix, so this squaring mapping gives the identity). Now instead of having nine different elements of the triticate representation of complex numbers in the exercise on chapter I (B), we have double that – eighteen.

The second is that a cyclic permutation, say $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ together with a twist of two elements, say $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, generates all permutations of three elements (the first generates, on its own, all cyclic permutations).

I want to ask a question. If we take the antidiagonal of a cyclic generator, that is, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ do we, with the diagonal and antidiagonal components always have for any polyticate number a set of group generators? Now if you wish, generate your own theories. In particular, I ask: what is the relationship between the nine elements of the first set of triticate numbers, the second extended set of 18 elements, and the fact that they use complex numbers, which have two components, a real and an imaginary part? If you have any conclusions, can they be generalised?

(G) *Look up the class number on the internet* (*the book by Conway and Guy*, *The book of numbers* [1CG00], *is also a good reference*). *What sort of divisors of these numbers* (*we might call them polyticate numbers*) *are there*?

Chapter II.

(A) *Obtain the inverse of the* J-*abelian number*

$(a1 + b\alpha + c\phi)_{(b1 + di + c\alpha)}$,

using intricate conjugates.

The intricate conjugate of a1 + bα + cϕ is a1 – bα – cϕ, and of b1 + di + cα is b1 – di – cα. Using the property of J-abelian numbers that $E_F G_H = (EG)_{(FH)}$, we find

$(a1 + b\alpha + c\phi)_{(b1 + di + c\alpha)}(a1 - b\alpha - c\phi)_{(b1 - di - c\alpha)}$
$= (a^2 - b^2 - c^2)1_{(b^2 + d^2 - c^2)1}$,

and since the top and bottom layers are both scalars

$= (a^2 - b^2 - c^2)(b^2 + d^2 - c^2)1_1$.

Thus to find $E_F G_H = 1_1$, G_H is the inverse of E_F, so that the inverse of

$(a1 + b\alpha + c\phi)_{(b1 + di + c\alpha)}$

is

$[(a^2 - b^2 - c^2)(b^2 + d^2 - c^2)]^{-1}(a1 - b\alpha - c\phi)_{(b1 - di - c\alpha)}$.

(B) *Using example* (A), *what is its determinant by this method*? *Check that it corresponds with the determinant of section* 15, *equation* (3).

The block representation of $(a1 + b\alpha + c\phi)_{(b1 + di + c\alpha)}$ given in section 2.15 is

$\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$,

and its determinant by equation (3), which can also be obtained using equation (2) of section 2.15 is

(det A)(det D),

so using the determinant formula for an intricate number from chapter I, section 1.6, this is

$(a^2 - b^2 - c^2)(b^2 + d^2 - c^2)$.

(C) *A* 4×4 *companion matrix is defined as*

$$C = \begin{bmatrix} 0 & 0 & 0 & -a_0 \\ 1 & 0 & 0 & -a_1 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & 1 & -a_3 \end{bmatrix}.$$

Show that the determinant

$$\det(C - xI) = \det\begin{bmatrix} -x & 0 & 0 & -a_0 \\ 1 & -x & 0 & -a_1 \\ 0 & 1 & -x & -a_2 \\ 0 & 0 & 1 & -x - a_3 \end{bmatrix}$$

is given by

$$g(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + x^4.$$

By equation (2) of 2.15

$$\det\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = (\det A')(\det(D' - C'A'^{-1}B')),$$

where in this case

$$A' = \begin{bmatrix} -x & 0 \\ 1 & -x \end{bmatrix} = -x1 - \frac{i}{2} + \frac{\phi}{2},$$

$$B' = \begin{bmatrix} 0 & -a_0 \\ 0 & -a_1 \end{bmatrix} = -\frac{a_1 1}{2} - \frac{a_0 i}{2} + \frac{a_1 \alpha}{2} - \frac{a_0 \phi}{2},$$

$$C' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{i}{2} + \frac{\phi}{2},$$

$$D' = \begin{bmatrix} -x & -a_2 \\ 1 & -x - a_3 \end{bmatrix} = -\left(x1 + \frac{a_3 1}{2}\right) - \frac{a_2 i}{2} + \frac{a_3 \alpha}{2} - \frac{a_2 \phi}{2},$$

so that

$$A'^{-1} = \begin{bmatrix} -x & 0 \\ 1 & -x \end{bmatrix}^{-1} = \frac{\left(-x1 + \frac{i}{2} - \frac{\phi}{2}\right)}{x^2 + \frac{1}{4} - \frac{1}{4}},$$

giving with a bit of work

$$\det(D' - C'A'^{-1}B') = \left(x^2 + a_3x + a_2 + \frac{a_1x + a_0}{x^2}\right),$$

which directly implies the result.

Chapter III.

(A) *Show that a quaternion represented in section* 3.7 *by* $a1_1 + bi_1 + c\alpha_i + d\phi_i$ *has determinant* $[a^2 + b^2 + c^2 + d^2]^2$, *where this is the square of the denominator in the equation for the inverse* 3.7.(2).

The determinant of a quaternion

$$a1_1 + bi_1 + c\alpha_i + d\phi_i$$

is given by

$$\det\begin{bmatrix} a & c & b & d \\ -c & a & -d & b \\ -b & d & a & -c \\ -d & -b & c & a \end{bmatrix},$$

which by formula 2.8.(2) may be expanded out in terms of the elements of the first row to

$$a \det\begin{bmatrix} a & -d & b \\ d & a & -c \\ -b & c & a \end{bmatrix} - c \det\begin{bmatrix} -c & -d & b \\ -b & a & -c \\ -d & c & a \end{bmatrix} + b \det\begin{bmatrix} -c & a & b \\ -b & d & -c \\ -d & -b & a \end{bmatrix} - d \det\begin{bmatrix} -c & a & -d \\ -b & d & a \\ -d & -b & c \end{bmatrix},$$

so that continuing these row expansions with determinants of 3 × 3 matrices, we get

$$\begin{aligned} &a[a(a^2 + c^2) + d(da - cb) + b(dc + ab)] \\ &- c[-c(a^2 + c^2) + d(-ba - cd) + b(-bc + ad)] \\ &+ b[-c(da - cb) - a(-ba - cd) + b(b^2 + d^2)] \\ &- d[-c(dc + ab) - a(-bc + ad) - d(b^2 + d^2)] \\ &\qquad = [a^4 + b^4 + c^4 + d^4 + 2a^2b^2 + 2a^2c^2 + 2a^2d^2 + 2b^2c^2 + 2b^2d^2 + 2c^2d^2] \\ &\qquad = [a^2 + b^2 + c^2 + d^2]^2. \end{aligned}$$

Chapter IV.

(A) *Let* X *and* Y *be symmetric matrices and* A *and* B *be antisymmetric matrices*. *Show*

$$\begin{aligned} &(X + A)(Y + B) + (X - A)(Y + B) + (X + A)(Y - B) + (X - A)(Y - B) = 4XY, \\ &(X + A)(Y + B) + (X - A)(Y + B) - (X + A)(Y - B) - (X - A)(Y - B) = 4XB, \\ &(X + A)(Y + B) - (X - A)(Y + B) + (X + A)(Y - B) - (X - A)(Y - B) = 4AY, \\ &(X + A)(Y + B) - (X - A)(Y + B) - (X + A)(Y - B) + (X - A)(Y - B) = 4AB. \end{aligned}$$

Express the above relations in terms of the transposes $(A + Y)^T$ *and* $(Y + B)^T$.

For a typical example

$$(X + A)(Y + B) + (X + A)^T(Y + B) + (X + A)(Y + B)^T + (X + A)^T(Y + B)^T = 4XY.$$

Reformulate these in terms of the matrix product, the left matrix product and transposes.

$$\begin{aligned} &(X + A)(Y + B) + (X + A)_{LT}(Y + B) \\ &\qquad + [(Y + B)_{LT}(X + A)]^T + [(Y + B)(X + A)]^T = 4XY, \text{ etc.} \end{aligned}$$

(B) *Using a* 2 × 2 *matrix in its intricate representation or otherwise, show that the product of two symmetric matrices is not necessarily symmetric*.

Consider the symmetric matrix

$$a1 + c\alpha + d\phi.$$

In general

$$\begin{aligned} &(a1 + c\alpha + d\phi)(a'1 + c'\alpha + d'\phi) \\ &\qquad = (aa' + cc' + dd')1 + (cd' - c'd)i + (ac' + a'c)\alpha + (ad' + a'd)\phi \\ &\qquad \neq (aa' + cc' + dd')1 - (cd' - c'd)i + (ac' + a'c)\alpha + (ad' + a'd)\phi \\ &\qquad = (a'1 + c'\alpha + d')(a1 + c\alpha + d\phi). \end{aligned}$$

(C) *Prove the Jacobi identity*

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0.$$

The definition of [A, B] is AB – BA, so

$$\begin{aligned} [[A, B], C] &= ((AB - BA)C - C(AB - BA)) \\ &= ABC - BAC - CAB + CBA. \end{aligned}$$

It follows that

$$\begin{aligned} &[[A, B], C] + [[B, C], A] + [[C, A], B] = (ABC - BAC - CAB + CBA) \\ &\qquad + (BCA - ABC - CBA + ACB) + (CAB - BCA - ACB + BAC) = 0. \end{aligned}$$

Chapter V.

(A) *This exercise is computationally exhausting on paper. An equation-solver might be more efficient. In equations* (6) *to* (15) *of section* 4, *put* a = p = 0, b = b, c = d = b′ = 1, c′ = d′ = b″ = c″ = d″ = 0. *Including the value of* b, *what is the solution of equation* (5)?

The result could be determined by working collectively in your group, which for life is more useful to know than just solving problems.

Chapter VI.

(A) *What is* φ(s) *when* s *is prime*?

From the definition

$$\varphi(s) = s\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right)\ldots\left(1 - \frac{1}{r}\right),$$

with s the product of primes p, q, ... r, to powers j, k, ... m respectively, so

$$\varphi(p) = p - 1.$$

(B) *Assume for natural numbers* n, s ∈ ℕ *that*

$$\varphi(s)[y^{\varphi(s)+1} - y] \equiv 0 \pmod{s}. \quad (1)$$

Prove by induction that

$$\varphi(s)[y^{n\varphi(s)+1} - y] \equiv 0 \pmod{s}. \quad (2)$$

By induction we mean, prove a suitable starting example (*say* n = 1), *assume* (2) *holds for* n, *and then prove for* (n + 1).

Note that you cannot assume if (1) *holds that either* φ(s)[y] = 0 or $y^{\varphi(s)} - 1 \equiv 0$ (mod s), *since say for* (mod 6), 2 × 3 = 0, *but* 2 ≠ 0 (mod 6) *and* 3 ≠ 0 (mod 6). *However, you could make such an assumption when s is prime*.

Equation (2) above holds for n = 1, which corresponds with (1), so that

$$\varphi(s)\, y^{n\varphi(s)}[y^{\varphi(s)+1} - y] \equiv 0 \pmod{s}, \quad (3)$$

and adding (2) and (3) provides the proof by induction for n + 1.

(C) *Show for a matrix* W *with a symmetric part* U *and an antisymmetric part* V, *giving*

W = U + V,

that, say, a trailing layer of p1 + qα + rϕ *can be applied to* U *which keeps its symmetric state, and a trailing layer of* ti *can be appended to* V *which makes the matrix also symmetric*.

A matrix is symmetric if and only if its basis elements are symmetric. We saw in chapter II, section 5, that a symmetric matrix has an even number of i's in all basis elements, and an antisymmetric matrix has an odd number of i's in all basis elements.

Using the results indicated above from chapter XI, *show that there exists an orthogonal matrix* Q (*so that the transpose* $Q^T = Q^{-1}$) *with*

$$\varphi(s)Q[(U_{p1 + q\alpha + r\phi} + V_{ti})^{n\varphi(s)+1} - (U_{p1 + q\alpha + r\phi} + V_{ti})]Q^{-1} \equiv 0 \pmod{s},$$

provided Q *and thus* Q^{-1} *are expressible* (mod s).

The result follows from chapter XI, section 2, on Sylvester's law of inertia for symmetric matrices. Q^{-1} exists, which has whole number entries (mod s) – note that rational numbers are integers (mod p), and this is in Q^T, by definition of an orthogonal matrix.

Chapter VII.

(A) *Using the strict transfer principle, show*

$$\sum_{\text{all } n \in \mathbb{N}} (n) = \tfrac{1}{2}\mathbf{\Omega}_{\mathbb{N}}(\mathbf{\Omega}_{\mathbb{N}} + 1)$$

and

$$\prod_{\text{all } n \in \mathbb{N}} (n) = \mathbf{\Omega}_{\mathbb{N}}!$$

This follows on applying the strict transfer principle to

$$S = \sum_{i=1}^{n}(i) = \frac{1}{2}n(n+1),$$

since

$$2S = [1 + 2 + ... + n] + [n + (n - 1) + ... + 1],$$

and the sums in pairs (1 + n), (2 + (n – 1)), ... (n + 1) all have value (n + 1).

Likewise we apply the strict transfer principle to the factorial function defined as

$$1.2. ... .n = n!$$

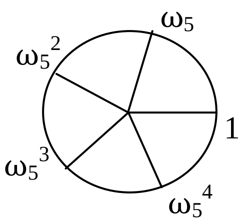
Chapter X.

(A) *Let* ω_5 *be the fifth root of unity. Show that the mappings*

$$\omega_5 \leftrightarrow \omega_5^3$$
$$\omega_5^2 \leftrightarrow \omega_5^4$$
$$1 \leftrightarrow 1$$

do not collectively form an automorphism.

Automorphisms of complex numbers are reflections on imaginary symbols. Collectively, they are of the same type.



The mapping 1 → 1 is a horizontal reflection. The horizontal reflection of ω_5^2 is ω_5^3 and of ω_5 is ω_5^4, and this is unique.

(B) *Show that a complex ring automorphism splits a pair of solutions into a '*real*' part with the same properties as if it were real, and a '*complex*' part with the same properties as if it were complex.*

Let symbols a and b be treated as real (the extension to complex numbers is not difficult). Let

$$z^2 + az + b = 0. \tag{1}$$

Put

$$z = x + iy, \tag{2}$$

then

$$x^2 - y^2 + ax + b + 2yxi + axi = 0, \tag{3}$$

so under the automorphism i → -i

$$x^2 - y^2 + ax + b - 2yxi - axi = 0, \tag{4}$$

and adding (3) and (4)

$$x^2 - y^2 + ax + b = 0, \tag{5}$$

whereas subtracting (3) from (4) gives

$$2yxi + axi = 0, \tag{6}$$

which is the same result as equating real and imaginary parts.

Show that a complex polynomial equation can be separated out into 'real' and 'complex' equations, which we have described as equating real and complex parts, so that there is a solution to a polynomial equation using ring automorphisms if and only if there is a solution equating real and complex parts. You need to prove a two-way implication here.

Consider isolated automorphisms. The proof above is generalised in a straightforward way. For each stage in the proof we can say the result follows if and only if the previous stage holds.

Chapter XI.

(A) *In equation* 11.5.(13) *for the quintic variety, we have so far only considered linear substitutions of variables. For the nonlinear allocation*

$$y = x^4 + rx^3 + sx^2 + tx + u,$$

on putting the linear substitutions

$$y = w + g,$$
$$x = w + h,$$

show that this is solvable, and thus in this case the nonlinear allocation reduces to a linear one.

The result of the substitution is a quartic in w, which we know from the results of chapter VIII, section 5, has a solution by radicals in multiplicative format. These four solutions are each in linear form.

(B) *Let*

$$x^3 + Kx^2 + Lx + M = 0, \quad (1)$$

adjoin by multiplication the root

$$(x + n) = 0. \quad (2)$$

Show by equating coefficients that there is no general such quartic polynomial equation with the result available by an inductive procedure equivalent to the solvable equation

$$(x^2 + px + q)^2 + a(x^2 + px + q) + b = 0. \quad (3)$$

If (1) and (2) hold then

$$x^4 + (K + n)x^3 + (Kn + L)x^2 + (Ln + m)x + Mn = 0.$$

Equation (3) is

$$x^4 + 2px^3 + (2q + p^2 + a)x^2 + p(2q + a)x + q^2 +qa + b = 0.$$

Hence on equating coefficients

$$K + n = 2p$$
$$Kn + L = 2q + p^2 + a$$
$$Ln + M = p(2q + a)$$
$$Mn = q^2 +qa + b,$$

giving

$$n = 2p – K,$$

and finally the cubic equation

$$p[K(2p – K) + l] = L(2p – K) + M + p^3,$$

so there is no inductive technique to obtain p.

(C) & (D). These are a series of questions. The reader is asked to follow long computations. For (C), there is a 'miraculous' cancellation, which it is very likely would not happen if the computation was incorrect, and for which I cannot see must follow other than from other arguments in the same chapter showing that the quintic is not solvable by radicals. If the reader can spot another reason for this, I think it would be interesting. For (D) I believe the calculation is correct.

Chapter XII.

(A) *Show the following logical operations are equivalence relations described in chapter* III, *section* 3: AND, OR *and* ⇔, *using* ⇔ *for* IF and only IF (*the truth table for which is given in section* 10).

An equivalence relation ≡ satisfies

m ≡ m
if m ≡ n then n ≡ m
if m ≡ n and n ≡ p then m ≡ p.

When ≡ is AND, this is only true for m and n when m is true and n is true. So if the subset of m, n and p referred to has the value true, AND is an equivalence relation, otherwise AND has the value false and again is an equivalence relation.

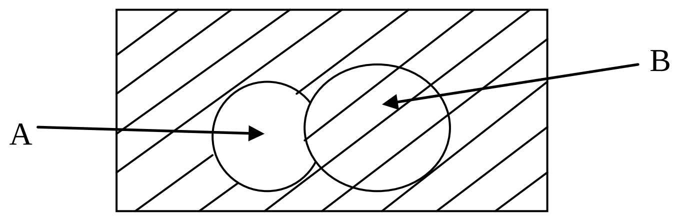
OR is only false for m and n when m is false and n is false. So if the subset of m, n and p referred to is false, OR is an equivalence relation, otherwise OR is true and again is an equivalence relation.

For ⇔, this is only true when m and n have the same truth values, otherwise it is false, and the same type of thinking applies.

(B) *For statements* A *and* B, A *implies* B *is written* A ⇒ B *and is only false when* A *is true and* B *is false*. *Using truth tables or otherwise*, *show* A ⇒ B *is the same as* (NOT A) OR B, *and* (A AND B) ⇔ A.

A ⇒ B	(A AND B) ⇔ A	(A AND B)	(NOT A) OR B	NOT A	A	B
T	T	T	T	F	T	T
T	T	F	T	T	F	T
F	F	F	F	F	T	F
T	T	F	T	T	F	F

We transform a statement A *to a set* A′ *through the identification of* A *with* a ∊ A′. *What are the corresponding Venn diagrams for* ‘implies’ *in sets*? *See the Prologue for a description of Venn diagrams*, *or look at the internet*.



In the diagram above, shaded = true, unshaded = false for ‘implies’

(x ∊ A) ⇒ (x ∊ B).

(C) *Show* ⇒ *defines a partial order of definition* 11.7.2, (*which is repeated in chapter* III *section* 5).

Check $A \Rightarrow A$,

  if $(A \Rightarrow B)$ & $(B \Rightarrow A)$ then $(A \Leftrightarrow B)$,

  if $(A \Rightarrow B)$ & $(B \Rightarrow C)$ then $(A \Rightarrow C)$.

(D) *Can you describe* 'includes' *as a Venn diagram*? *What is its corresponding mapping back to symbolic logic*? *Is this a partial order*? *Is the Venn diagram for* (B) *different from or just an extension of the Venn diagram for includes*?

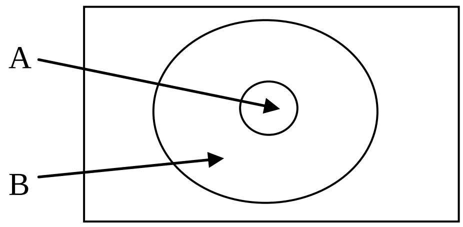


diagram for A is included in B

Includes satisfies if A does not include B then

  (x does not belong to A) AND (x belongs to B),

and thus is the diagram for $\Rightarrow$.

(E) A is sufficient for B *means* A implies B. A is necessary for B *means* A is implied by B (*the same as* B implies A). *What does the internet say about modal logics*? *Do you think it is a good or a bad idea to have* 'necessary' *and* 'sufficient' *as primitive terms*?

Modal logics are discussed in *Number, space and logic* [Ad18].

Chapter XIII.

(A) *A nuclear power plant has two processes. The probability of the first process failing is* $A = a\tau + (1 - a)\upsilon$, *and of the second failing is* $B = b\tau + (1 - b)\upsilon$. *There is a* $1/5^{th}$ *certainty that if* A *fails then* B *will. What is the probability that* A *and* B *will happen together*?

$$P(A \Rightarrow B) = (1 - a + ab)\tau + (a - ab)\upsilon$$
$$= (1/5)\tau + (4/5)\upsilon,$$

so $a = 4/[5(1 - b)]$ and

$$P(A \,\&\, B) = ab\tau + (1 - ab)\upsilon = \{4b/[5(1 - b)]\}\tau + \{1 - 4b/[5(1 - b)]\}\upsilon.$$

Chapter XIV.

Prove the step-down equation for right nest operators of section 14.3

$$\forall x, y, n \;\; x \,|^{n}\, (y + 1) = (x \,|^{n}\, y) \,|^{n-2}\, (\hat{x} \,|^{n}\, (y + 1)).$$

$$x \,|^{n-1}\, (x \,|^{n-1} \ldots (x \,|^{n-1}\, x))$$

for $(y + 1)$ terms is

$$x \,|^{n-1}\, (x \,|^{n-1} \ldots (x \,|^{n-1}\, (1 + (x - 1))))$$

which is

$$[x \,|^{n-1}\, (x \,|^{n-1} \ldots (x \,|^{n-1}\, 1))] \,|^{n-2}\, [x \,|^{n-1}\, (x \,|^{n-1} \ldots (x \,|^{n-1}\, (x - 1)))]$$

for $(y + 1)$ terms, or

$$[x \,|^{n}\, y] \,|^{n-2}\, [\hat{x} \,|^{n}\, (y + 1)]$$

which defines $\hat{x}$.

Chapter XVI.

Show for a positive number a *that*

$(a^b)^c = a^{(bc)}$,
$(a^{ib})^c = a^{(ibc)}$,
$(a^b)^{ic} = a^{(ibc)}$,
$(a^{ib})^{ic} = a^{(iwbc)}$,

holds in general if and only if the same equations hold with the exponential number e *replacing* a.

Show that this also holds when a *is negative if and only if it holds for* (-e).

For a positive, substitute $a = e^d$. Then for negative values -a = (-1)a, so that if the axioms hold for any a, they also hold for (-1) as a value of a.

Chapter XVII.

Develop a difference algebra which is the finite case of differentiation in section 17.11.

For further discussion, see *Number*, *space and logic* [Ad18].

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