CHAPTER V

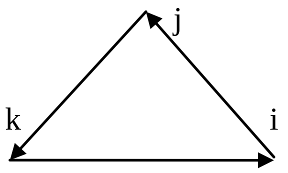
Novanions

5.1. Introduction.

Using homology theory J. F. Adams proved in 1960 that the only possible division algebras with a finite basis are at maximum those of the octonions, which are nonassociative, where quaternions and complex numbers are subalgebras of these. However, I introduce the explicit result that there exist nonassociative algebras, the n-novanions, with nonzero division, but when just the scalar part is zero two nonzero novanions may have a zero product. I obtain here the proof for n = 10, and extend it for n greater than 10. Novanions have a dimension of $n = 1 + 3^f \prod_{i \in \mathbb{N}} (3^{g_i} - 2)_i$, for which $f \in \mathbb{N}_{\cup 0}$, $g_i \in \mathbb{N}$. As a consequence, for the n-novanions I have opened a new type of quasi division algebra existence question, with novel results.

5.2. The quaternions and exquaternions.

The quaternions may be represented by a triangle diagram



where the nodes i, j and k satisfy

$$ij = k,\ jk = i,\ ki = j, \qquad (1)$$

in other words, for a positive sign in the above relations, we are following the arrows. When we are going in a direction opposite to the arrows, we have a negative sign:

$$ji = -k,\ kj = -i,\ ik = -j. \qquad (2)$$

We have here that 1 commutes with all elements, and also

$$1^2 = 1,\ i^2 = j^2 = k^2 = -1. \qquad (3)$$

Some matrix representations of the quaternions are given in chapter III, section 3.7. We now introduce the nonassociative exquaternions (which thus cannot be represented by a matrix).

The linearly independent intricate basis elements of chapter I, section 1.6 satisfy

$$1^2 = 1,\ i^2 = -1,\ \alpha^2 = 1,\ \phi^2 = 1,$$

$$1i = i = i1,\ 1\alpha = \alpha = \alpha 1,\ 1\phi = \phi = \phi 1,$$

$$i\alpha = -\phi = -\alpha i,\ i\phi = \alpha = -\phi i \text{ and } \alpha\phi = i = -\phi\alpha.$$

This algebra may be modified so that, for instance, all relations are maintained except for one case of the sign, which is altered to

$$i\alpha = \phi = -\alpha i.$$

If the resulting algebra were associative, then

$$\alpha = i\phi = i(i\alpha) = (i^2)\alpha = -\alpha$$

and

$$-i = \phi\alpha = (i\alpha)\alpha = i\alpha^2 = i,$$

so this *extricate* (in contradistinction to *intricate*) algebra is not associative, and thus is not represented by a matrix.

Indeed, we may adopt the relations

$$i(i\alpha) = -(i^2)\alpha$$

and

$$(i\alpha)\alpha = -i(\alpha^2).$$

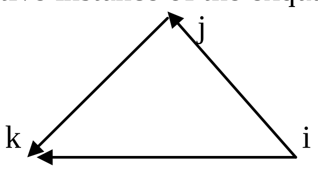
If we apply the quaternion representation of chapter III to the exquaternions, in the sense of an algebra rather than a matrix, then a representation is

$$e_1 = i_1,\ e_2 = \alpha_i,\ e_3 = \phi_i,$$

and the exquaternion multiplication table on elements becomes

×	***1***	***e₁***	***e₂***	***e₃***
1	1	e_1	e_2	e_3
$\boldsymbol{e_1}$	e_1	-1	e_3	e_2
$\boldsymbol{e_2}$	e_2	$-e_3$	-1	e_1
$\boldsymbol{e_3}$	e_3	$-e_2$	$-e_1$	-1

A representative instance of the exquaternions can be depicted in the diagram



where the direction of the arrows indicates the sign in products. For this particular example

$$(j - k)(1 - i) = 0, \qquad (4)$$

so that the exquaternions do not form a division algebra. □

We can combine the intricate and extricate algebras to form a *duplicate* algebra. For this purpose, it is convenient under the extricate algebra to introduce basis elements written as $i_\#$ (or i# in layers) for i, $\alpha_\#$ (or α#) for α, and $\phi_\#$ or ϕ# for ϕ. The general duplicate number is now written as

$$a1 + bi + c\alpha + d\phi + b'i_\# + c'\alpha_\# + d'\phi_\#.$$

We need a multiplication operation for such numbers. If A and B are intricate numbers, $A_\#$ and $B_\#$ extricate numbers, define

$$(A + A_\#) \times (B + B_\#) = (AB) + (A_\# B) + (A \times_\# B_\#) + (A_\# \times_\# B_\#),$$

where AB is expressed under intricate multiplication as an intricate number, $A_\# \times_\# B_\#$ under extricate multiplication $\times_\#$ as an extricate number, whereas the $A_\# B$ term is taken as an intricate multiplication with the $A_\#$ variables transformed to intricate ones, and $A \times_\# B_\#$ as extricate multiplication, with the A variables transformed to extricate ones.

We may also form the following general nonassociative algebra, for the set

$$\Lambda = \{\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3\},$$

where Λ_0, Λ_1, Λ_2 and Λ_3 are scalars and

$$(A + A_\#) \times_\Lambda (B + B_\#) = \Lambda_0(AB) + \Lambda_1(A_\# B) + \Lambda_2(A \times_\# B_\#) + \Lambda_3(A_\# \times_\# B_\#).$$

The hyperduplicate algebra is obtained by the same process that a hyperintricate algebra is obtained from an intricate one. In describing nonassociative operations the hyperduplicate algebra needs extension to represent a general multiplication table.

We will see later that there exist nonassociative algebras, which cannot be represented by matrices and are partial division algebras, the novanions. □

To demonstrate that the product of two quaternions with real coefficients

$$(a1 + bi + cj + dk)(p1 + qi + rj + tk) \quad (5)$$

cannot be zero unless a = b = c = d = 0 or p = q = r = t = 0, we will adopt a proof 'the long way round', which we will simplify and extend later to the nonassociative 10-novanions.

Equating by hypothesis the real and quaternionic parts of (5) to zero, we obtain the following 4 equations in 8 unknowns.

real part:

$$ap - bq - cr - dt = 0 \quad (6)$$

i part:

$$pb + aq + ct - dr = 0 \quad (7)$$

j part:

$$pc + ar - bt + qd = 0 \quad (8)$$

k part:

$$pd + at + br - cq = 0. \quad (9)$$

If a = 0, then equations (6), (7) (8) and (9) imply

$$(b^2 + c^2 + d^2)q = 0, \quad (10)$$

$$(b^2 + c^2 + d^2)r = 0 \quad (11)$$

and

$$(b^2 + c^2 + d^2)t = 0, \quad (12)$$

so either b = c = d = 0, which we exclude, or q, r and t = 0, which implies (5) cannot be satisfied except for b = c = d = 0.

On eliminating p, provided a ≠ 0 we obtain

$$(a^2 + b^2)q + (-ad + cb)r + (ac + db)t = 0, \quad (13)$$

$$(ad + cb)q + (a^2 + c^2)r + (-ab + cd)t = 0 \quad (14)$$

and

$$(-ac + bd)q + (ab + cd)r + (a^2 + d^2)t = 0. \quad (15)$$

On eliminating q,

$$[(a^2 + c^2)(a^2 + b^2) + (ad + cb)(ad - cb)]r + [(-ab + cd)(a^2 + b^2) - (ad + cb)(ac + db)]t = 0 \quad (16)$$

and

$$[(ab + cd)(a^2 + b^2) - (-ac + db)(-ad + cb)]r + [(a^2 + d^2)(a^2 + b^2) - (-ac + db)(ac + db)]t = 0. \quad (17)$$

Then on eliminating r, whilst putting

$$A = a^2,\ B = b^2,\ C = c^2,\ D = d^2, \quad (18)$$

we obtain provided p = q = r = t = 0 does not hold

$$[(A + C)(A + B) + (AD - CB)][(A + D)(A + B) - (-AC + DB)]$$
$$- [(A + B)^2(-AB + CD) + (-AD + CB)(-AC + DB)]$$
$$+ (-ab + cd)(A + B)(-ac + db)(-ad + cb) + (ab + cd)(A + B)(ad + cb)(ac + db)$$
$$= 0. \quad (19)$$

The last line with terms in (19) is

$$2(A + B)[BCD + ACD + ABD + ABC], \quad (20)$$

so we have

$$(A + B)^2A[A + B + C + D]$$
$$+ (A + B)[A^2D + AD^2 + A^2C + AC^2 + ABC + ACD + ABD] = 0 \quad (21)$$

which is impossible, since A > 0. ☐

5.3. The octonions and exoctonions.

The algebra of the nonassociative octonions, is given by the Fano plane in the figure below.

Typical cyclic identities are

$$e_7e_5 = e_2 \quad (1)$$

$$e_5e_6 = e_1$$

and

$$e_4e_2 = e_6.$$

Note what we have said here. The inner triple $e_2e_4e_6$ acts like a quaternion, but the outer triple $e_1e_3e_5$ does not. Nevertheless, we will need to allocate later a list ordered as right triple = $e_2e_4e_6$ + central triple = $e_1e_3e_5$ + one = e_7. Octonions form a division algebra, in particular

$$e_c^2 = -1, \quad (2)$$

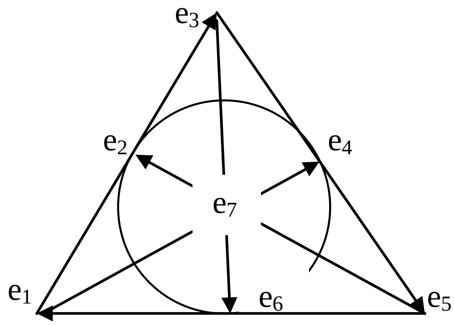
$$e_ae_b = -e_be_a \ (a \neq b),$$

and the inverse of

$$a1 + \sum_{n=1}^{7} b_n e_n$$

is

$$(a1 - \sum_{n=1}^{7} b_n e_n)/(a^2 + \sum_{n=1}^{7} b_n^2). \quad (3)$$



The octonions, $\mathbb{O}$, are also generated by the Cayley-Dickson construction [Ba01]. This builds up algebras from the complex numbers, to the quaternions, to the octonions, to the sixteen dimensional sedenions, etc.

In the intricate representation there are two types of mappings we now consider, a number to its negative, or a number to its transpose, denoted with a T. We have

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^T = i^T = -i,$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^T = \alpha^T = \alpha,$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^T = \phi^T = \phi.$$

To generate an octonion from the quaternions, we consider a pair of quaternions in labelled brackets $(K)_1$, $(L)_2$, with

$$(K)_1 = (a1_1 + bi_1 + c\alpha_i + d\phi_i)_1,$$

$$(L)_2 = (a'1_1 + b'i_1 + c'\alpha_i + d'\phi_i)_2.$$

To give an algebra for this, to add two ordered pairs, add their components. To multiply two ordered components, $(A)_1(B)_1$ is a quaternion $(AB)_1$, $(C)_2(D)_2$ is the quaternion $(-CD^T)_1$, whereas $(A)_1(C)_2$ is the quaternion $(-AC^T)_2$. Note that in the second set

$$(1_1)_2(1_1)_2 = (-1_1)_1.$$

We will identify this octonion basis element as the element e_7 at the centre of the Fano plane.

Define a T-algebra to be an algebra equipped with conjugation, a linear map T satisfying

$$a^{TT} = a, \quad (4)$$

$$(ab)^T = b^T a^T. \quad (5)$$

Starting from any T-algebra, the Cayley-Dickson construction gives a new algebra

$$(a, b)(c, d) = (ac - db^T, ad^T + cb), \quad (6)$$

with conjugation defined by

$$(a, b)^T = (a^T, -b). \quad (7)$$

This generates the basis element multiplication table $\mathbb{O} \times \mathbb{O} \to \mathbb{O}$ of the octonions

×	***1***	***e₁***	***e₂***	***e₃***	***e₄***	***e₅***	***e₆***	***e₇***
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

We can generate for each

$$\mathbb{O} \times \mathbb{O} \to \mathbb{O}: e_i \times e_j \to e_k$$

a Cayley-Dickson construction of

$$e_i \times (-e_j) \to -e_k,$$

so that each of the 7 non-scalar basis elements in a row of the table can be multiplied by -1 to provide $2^7 = 128$ possible Cayley-Dickson constructions.

The Fano plane has 7 non-scalar basis elements. The number of non-scalar quaternionic triplets is 7, each of which, since exquaternions are excluded, operates under a forward or a reversed orientation – again 2^7 possibilities. The following Fano triplets map bijectively to the standard Cayley-Dickson construction for the octonions

$$(e_1, e_2, e_3), (e_3, e_4, e_5), (e_1, e_4, e_6), (e_4, e_6, e_2), (e_1, e_7, e_6), (e_4, e_7, e_3), (e_5, e_7, e_2).$$

Thus distinct possibilities for the Fano plane are isomorphic to distinct instances of the Cayley-Dickson construction. ☐

A Fano plane can be devised to form the exoctonions. Representing the exquaternions of section 2 by the 4 × 4 block A, the exoctonion multiplication table may be represented by the blocks

A B
C D

where A or its transpose A^T may be chosen and D or its transpose D^T. An example of the complex numbers, exquaternions and exoctonions is represented in the following nested table for an instance of the exoctonions.

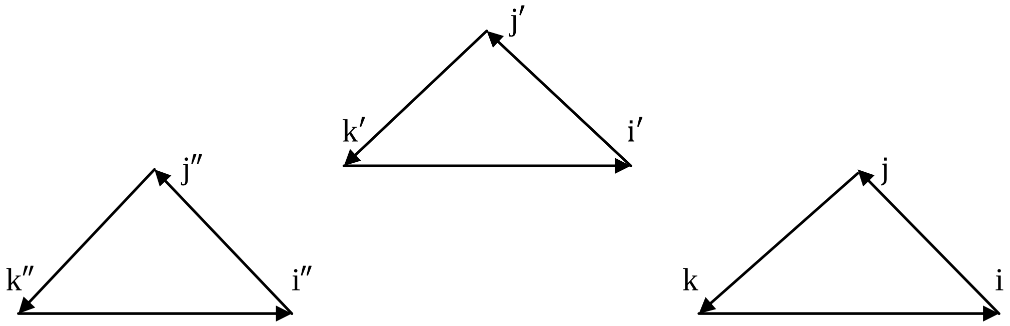
×	***1***	***e₁***	***e₂***	***e₃***	***e₄***	***e₅***	***e₆***	***e₇***
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	e_2	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	$-e_2$	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	e_3	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	$-e_3$	-1	e_1
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	$-e_1$	-1

The exoctonions do not form a division algebra, since they contain the exquaternions as a subalgebra, but the inverse of $a_0 1 + a_1 e_1 + a_2 e_2 + \ldots$ etc. exists and is

$$(a_0 1 - a_1 e_1 - a_2 e_2 - \ldots \text{ etc.})/(a_0^2 + a_1^2 + a_2^2 + \ldots \text{ etc.}). \ \square$$

5.4. The 10-novanions.

We now introduce the 10-novanions, represented by the set of triangle diagrams



where in general each triangle is a quaternion without 1.

The primed variables ( ), (′) and (″) act as holders of information concerning an algebra for them. When the variables all contain a common instance, for example (k), (k′) and (k″), then the algebra is that of the quaternions, in which we have a cyclic algebra

$$kk' = k'' = - k'k. \qquad (1)$$

When the variables contain different instances, such as k and i′, then the product contains the primed variable that does not belong to the first two elements, but the primed part commutes.

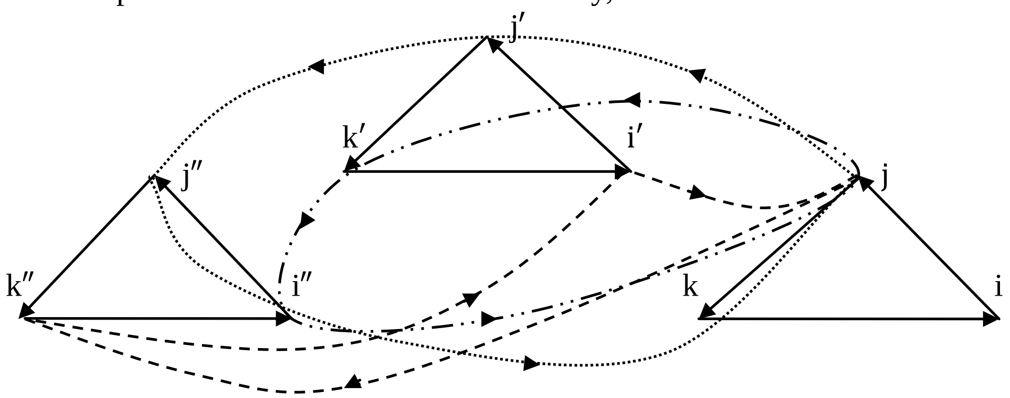
On top of this is the fact that the variables, say k and i, satisfy a quaternion algebra, so say

$$ki = j = -ik \qquad (2)$$

and consequently

$$ki' = j'' = -i'k. \qquad (3)$$

In order to picture the 10-novanions more closely, we will show the connections from node j



Our claim is that the inverse of

$$a1 + \sum_{n=1}^{3} \sum_{\text{primed } m=1}^{3} b_n^m e_n$$

is

$$(a1 - (\sum_{n=1}^{3} \sum_{\text{primed } m=1}^{3} b_n^m e_n))/(a^2 + (\sum_{n=1}^{3} \sum_{\text{primed } m=1}^{3} (b_n^m)^2)), \qquad (4)$$

and this constitutes a type of division algebra with no divisors of zero provided a1 ≠ 0 – the 10 dimensional 10-novanions.

We see that the n-novanions are nonassociative, since they have more than 4 basis elements; more explicitly

$$(j''k')j' = ij' = k'' \neq j''(k'j') = -j''i' = k.$$

We wish to enquire under what conditions there exist two 10-novanion numbers multiplied together giving zero:

$$(a1 + bi + cj + dk + b'i' + c'j' + d'k' + b''i'' + c''j'' + d''k'') \times (p1 + qi + rj + tk + q'i' + r'j' + t'k' + q''i'' + r''j'' + t''k'') = 0. \quad (5)$$

Their product is

<u>real part</u>:

$$ap - bq - cr - dt - b'q' - c'r' - d't' - b''q'' - c''r'' - d''t'' = 0, \quad (6)$$

<u>i part</u>:

$$bp + aq - dr + ct - b''q' - d''r' + c''t' + b'q'' - d'r'' + c't'' = 0, \quad (7)$$

<u>j part</u>:

$$cp + dq + ar - bt + d''q' - c''r' - b''t' + d'q'' + c'r'' - b't'' = 0, \quad (8)$$

<u>k part</u>:

$$dp - cq + br + at - c''q' + b''r' - d''t' - c'q'' + b'r'' + d't'' = 0, \quad (9)$$

<u>i′ part</u>:

$$b'p + b''q - d''r + c''t + aq' - d'r' + c't' - bq'' - dr'' + ct'' = 0, \quad (10)$$

<u>j′ part</u>:

$$c'p + d''q + c''r - b''t + d'q' + ar' - b't' + dq'' - cr'' - bt'' = 0, \quad (11)$$

<u>k′ part</u>:

$$d'p - c''q + b''r + d''t - c'q' + b'r' + at' - cq'' + br'' - dt'' = 0, \quad (12)$$

<u>i″ part</u>:

$$b''p - b'q - d'r + c't + bq' - dr' + ct' + aq'' - d''r'' + c''t'' = 0, \quad (13)$$

<u>j″ part</u>:

$$c''p + d'q - c'r - b't + dq' + cr' - bt' + d''q'' + ar'' - b''t'' = 0, \quad (14)$$

<u>k″ part</u>:

$$d''p - c'q + b'r - d't - cq' + br' + dt' - c''q'' + b''r'' + at'' = 0. \quad (15)$$

D is a possibly nonassociative division algebra if for any element a in D and any nonzero element b in D there exists just one element x in D with a = bx and only one element y in D with a = yb.

If a = 0, the 10-novanions contain possibilities for two nonzero 10-novanions giving a product which is zero. We give an example due to Doly García, showing that the 10-novanions satisfying a = 0 do not form a division algebra of standard type

$$(i + i' + i'')(j + j' - 2j'') = 0. \quad (16)$$

Thus the 10-novanions are not a division algebra, since for an arbitrary real number g

$$(i + i' + i'')[(j + j' - 2j'')g] = 0. \quad (17)$$

From now on we will assume a ≠ 0. By a symmetrical argument applied also to the theorem which follows, we need to assume with this that p ≠ 0.

Equations (6) to (15) form a matrix E + aI, where E is an antisymmetric matrix and I is the unit diagonal, multiplied on the right by the eigenvector (p, q, r, t, q′, r′, t′, q″, r″, t″). Below we give a proof that the eigenvalues of a real antisymmetric matrix are entirely imaginary, provided in chapter 11 of [Uh01], so these correspond to -a, which is real, whereas we are now excluding the only possibility for this, a = 0. □

Within the field of complex numbers ℂ, the complex conjugate of c = a + ib is c* = a – ib. For the corresponding matrix C with entries $c_{jk} = a_{jk} + ib_{jk}$, the conjugate $C^* = a_{jk} - ib_{jk}$. The transpose of a matrix C is denoted by C^T and has entries c_{kj}. The transpose is a contravariant (order reversing) operation:

$$(CD)^T = D^T C^T.$$

A matrix is defined as antisymmetric if $C^T = -C$.

Theorem: *All eigenvalues of a real antisymmetric matrix* $E = -E^T$ *are pure imaginary*.

Proof. Consider the case of the eigenvalue λ and possibly complex eigenvector $\mathbf{x} \neq \mathbf{0}$. According to the Fundamental Theorem of Algebra λ ∈ ℂ. Hence

$$E\mathbf{x} = \lambda\mathbf{x}. \qquad (18)$$

If we take the complex conjugate of both sides of the eigenvalue-eigenvector equation (18), we obtain

$$(E\mathbf{x})^* = (\lambda\mathbf{x})^* = \lambda^*\mathbf{x}^*.$$

Transposing yields

$$(E\mathbf{x})^{*T} = x^{*T}E^{*T} = \lambda^*\mathbf{x}^{*T}.$$

Define the norm

$$||E\mathbf{x}||^2 = (\lambda^*)\lambda(\mathbf{x}^{*T})\mathbf{x}, \qquad (19)$$

where

$$||\mathbf{x}|| = \sqrt{\mathbf{x}^T\mathbf{x}} \in \mathbb{R}.$$

Since $E^T = -E$ and $E^{*T} = E^T$ for a real antisymmetric matrix E, we can write (19) as

$$\begin{aligned} ||E\mathbf{x}||^2 &= \mathbf{x}^{*T}E^{*T}\, E^T\mathbf{x}, \\ &= \mathbf{x}^T E^T E\mathbf{x} \\ &= -\mathbf{x}^T E^2\mathbf{x} \\ &= -\mathbf{x}^T\lambda^2\mathbf{x}, \end{aligned}$$

because $E^2\mathbf{x} = E(E\mathbf{x}) = E(\lambda\mathbf{x}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}$, so

$$||E\mathbf{x}||^2 = -\lambda^2\mathbf{x}^T\mathbf{x}. \qquad (20)$$

Now $\mathbf{x}^T\mathbf{x} \neq 0$, and thus $\lambda^*\lambda = -\lambda^2$, by comparing (19) and (20). Thus a real antisymmetric matrix $E = -E^T$ can only have imaginary eigenvalues λ. □

5.5. n-novanions.

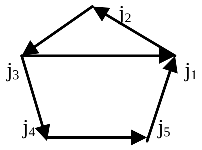
The octonions given by a Fano plane under suitable orientations may be given three copies in primed variables ( ) , (′) and (″), and by an analogous procedure this constitutes a 1 + (3 × 7) = 22 dimensional division algebra. Further details will be given in sections 5.6 and 5.7.

Extending these ideas further to multiple occurrences of the three or seven primed variables, say (′), (′)′ and (′)″, we obtain in general an $n = 1 + 3^f7^g$ dimensional novanion algebra, the n-novanions, in which if a common variable, k, is employed, then the lowest value within brackets of say (k), (k′)′ and (k″)″ is evaluated. □

Note that, just as we introduced the exquaternions, in a perfectly analogous fashion we can assemble exnovanions. □

5.6. The search for other novanion algebras.

Are there other novanion algebras of a type not already covered? This question has been stimulated by a first-year Sussex University student's identification of novanions with strings in physics, [Ad17], for which we wish to investigate the bosonic allocation $1 + 5^2$. In the pentagonal diagram shown next, an initial attempt depicts only one out of five subtriangles.



The pentagon can be enumerated cyclically, so that

$$j_1j_2 = j_3,\ j_2j_3 = j_4,\ j_3j_4 = j_5,\ j_4j_5 = j_1,\ j_5j_1 = j_2, \qquad (1)$$

and jumping a vertex we evaluate the closest triangle

$$j_3j_1 = j_2,\ j_4j_2 = j_3,\ j_5j_3= j_4,\ j_1j_4 = j_5,\ j_2j_5 = j_1, \qquad (2)$$

where on inverting the orientation, we get a minus sign.

This latter fact implies we have an inbuilt norm and inverse; the inverse of

$$a1 + \sum_{n=1}^{5} b_n j_n$$

is

$$a1 - \sum_{n=1}^{5} b_n j_n/(a^2 + \sum_{n=1}^{5} b_n^2), \qquad (3)$$

which is nonassociative, as is demonstrated by

$$(j_3j_1)j_4 = -j_3 \neq j_3(j_1j_4) = -j_4.$$

The question arises as to whether this constitutes a novanion algebra, which would now be extended from previous considerations to include the dimensions

$$n = 1 + 3^f5^g7^h.$$

A general matrix H may be represented as a sum of an antisymmetric part H_{anti} and a symmetric part H_{sym}. Then H_{sym} and H_{anti} are linearly independent over real coefficients, meaning there exist no real numbers c and d satisfying

$$cH_{sym} + dH_{anti} = 0.$$

By a demonstration analogous to that in section 4, and proved directly in [Uh01], the eigenvalues of a symmetric matrix are real. Further, a complex number with real coefficients h_1 and h_2 is linearly independent between h_1 and h_2i over real coefficients. It now follows that the eigenvalue equation

$$H = hI \qquad (4)$$

which has a unique set of n solutions is satisfied by

$$H_{sym} = h_1I$$

with n solutions and

$$H_{anti} = h_2I,$$

also with n solutions. These possible solutions are the only ones, since the solution set of (11) is unique. So if H_{sym} is not the zero matrix, a real h_1 exists. This implies there is no novanion algebra available.

The possibility of the existence of the division algebra violating equation

$$(a1 + bj_1 + cj_2 + dj_3 + ej_4 + fj_5) \times (p1 + qj_1 + rj_2 + tj_3 + uj_4 + vj_5) = 0 \qquad (5)$$

will now be investigated.

Under the constraints (1) and (2) we obtain the set of equations

real part:

$$ap - bq - cr - dt - eu - fv = 0, \tag{6}$$

j_1 part:

$$bp + aq - fr + 0 - fu + (c + e)v = 0, \tag{7}$$

j_2 part:

$$cp + (f + d)q + ar - bt + 0 - bv = 0, \tag{8}$$

j_3 part:

$$dp - cq + (b + e)r + at - cu + 0 = 0, \tag{9}$$

j_4 part:

$$ep + 0 - dr + (c + f)t + au - dv = 0, \tag{10}$$

j_5 part:

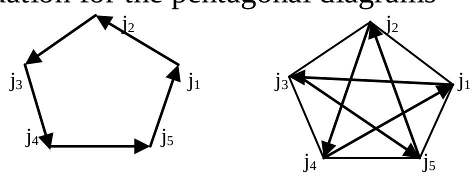
$$fp - eq + 0 - et + (d + b)u + av = 0, \tag{11}$$

from which it follows that the E type matrix is not antisymmetric, but it may be represented as the sum of two matrices F and G, where F has all pure imaginary eigenvalues:

$$F = \begin{bmatrix} 0 & -b & -c & -d & -e & -f \\ b & 0 & -f & c & -f & e \\ c & f & 0 & -b & d & -b \\ d & -c & b & 0 & -c & e \\ e & f & -d & c & 0 & -d \\ f & -e & b & -e & d & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c & 0 & c \\ 0 & d & 0 & 0 & -d & 0 \\ 0 & 0 & e & 0 & 0 & -e \\ 0 & -f & 0 & f & 0 & 0 \\ 0 & 0 & -b & 0 & b & 0 \end{bmatrix},$$

and we do not have pure imaginary eigenvalues for $F + G - \lambda I$. □

We find a similar type of situation for the pentagonal diagrams



since the diagram for j_1, j_2, j_3 is unoriented and therefore does not constitute a quaternion, or in fact a division algebra, so we have failed on modifying equations (5) to (10) to come up with other solutions, where for pentagonal diagrams we can show these pentagons consist in the general case of combined oriented and unoriented quaternion diagrams. □

However, the identification relates to the number 1 + 25, and we now wish to probe the allocation 25 = 7 + 9 + 9, where 7 is the number of non-real basis elements of the octonions, and 9 is the number for the 10-novanions. There is an analogy here. The octonion non-real basis elements of 7 may be represented as 1 + 3 + 3, where 3 is the number of such basis elements for the quaternions, and 1 for the complex numbers. We are forced for a number of reasons to decompose such an allocation into triplets, basically to retain the cyclic algebra for the quaternions.

The allocation will be as follows, where we subscript 3 and 1 to distinguish them

$3_a, 3_b, 3_c$ (i)

$3_d, 3_e, 3_f$ (ii)

$3_g, 3_h, 1_u$ (iii)

where allocations (i) and (ii) are internally similar to 10-novanions, and allocation (iii) is internally an octonion. We will explain why we use the word ‘similar’ later.

There are a number of possible configurations.

We want an algebra linking between (i), (ii) and (iii). Vertical allocations are present. We will choose next from straight lines going from left to right, for example the diagonal going upwards from 3_g, 3_e to 3_c. This is similar to a 10-novanion algebra. The descending line from 3_a, 3_e to 1_u is an octonion algebra. We then incorporate the algebra taking for example 3_d, 3_b to 1_u, an octonion algebra, or 3_g, 3_b to 3_f, this is similar to a 10-novanion algebra.

We have used the words 'similar to a 10-novanion algebra', and we now explain why. If we look at allocation (iii), this is part of the 3_g 3_h 1_u octonion, where 3_h and 1_u are linked. Although 3_g is indeed a quaternion, we have already mentioned that 3_h is not. Therefore the vertical allocation given by 3_a 3_d 3_g is a 10-novanion, since it is made of genuine quaternions, but the vertical allocation 3_b 3_e 3_h is not. 3_b, 3_e and 3_h occur in octonion representations. If we were to state that the central triple 3_b, 3_e and 3_h algebras were quaternions, we would have an inconsistency. Therefore for these allocations as part of a 'similar to 10-novanion' structure, we decide that the octonion structure overrides the 10-novanion one. Since there is only one special 1_u part for the octonions, this part of the allocation is unique. The similar 10-novanion structure is now not a closed algebra within the 10-novanions; part of it belongs to the octonions. The corresponding situation just for 10-novanions with no octonionic overlap but with novanionic overlap is described by the octonionic allocation already discussed.

Since there is no other mixing of allocations, the result is as consistent as the 10-novanions and the octonions. This can be checked with equations like 5.4.(6) to (15), for which it is clear eigenvalues are pure imaginary. Finally a calculation like 5.4.(16) shows that this is a novanion algebra. □

The existence of 10-, 26- and 80-novanions (the latter obtained by an array cube of items like (i) to (iii) – all configurations lie in planes of the cube, and an m-cube gives rise to a $(3^m \pm 1)$-novanion) implies that results derived for division algebras have a different extension for n-novanions. □

5.7. Sedenions and 64-novanions.

The 16-dimensional sedenions are formed by the Cayley-Dickson construction [Ba01]. Since they are not alternative, they do not form a division algebra. That is, we do not have

$$x(xy) = (xx)y$$

and

$$(yx)x = y(xx)$$

for all x and y in the algebra. Every associative algebra is alternative, but so too are some strictly non-associative algebras such as the octonions. The proof for octonions can be given exhaustively using basis elements.

For the example table that follows

$$(e_1 + e_{10})(e_{15} - e_4) = 0. \square$$

The standard complex numbers, quaternions, octonions and sedenions can be nested by inclusion and an instance is represented in the following table for the sedenions

×	***1***	$\boldsymbol{e_1}$	$\boldsymbol{e_2}$	$\boldsymbol{e_3}$	$\boldsymbol{e_4}$	$\boldsymbol{e_5}$	$\boldsymbol{e_6}$	$\boldsymbol{e_7}$	$\boldsymbol{e_8}$	$\boldsymbol{e_9}$	$\boldsymbol{e_{10}}$	$\boldsymbol{e_{11}}$	$\boldsymbol{e_{12}}$	$\boldsymbol{e_{13}}$	$\boldsymbol{e_{14}}$	$\boldsymbol{e_{15}}$
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
$\boldsymbol{e_1}$	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6	e_9	$-e_8$	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$
$\boldsymbol{e_2}$	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$	e_{10}	e_{11}	$-e_8$	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}
$\boldsymbol{e_3}$	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$	e_{11}	$-e_{10}$	e_9	$-e_8$	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}
$\boldsymbol{e_4}$	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3	e_{12}	e_{13}	e_{14}	e_{15}	$-e_8$	$-e_9$	$-e_{10}$	$-e_{11}$
$\boldsymbol{e_5}$	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	$-e_8$	e_{11}	$-e_{10}$
$\boldsymbol{e_6}$	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	$-e_8$	e_9
$\boldsymbol{e_7}$	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	$-e_8$
$\boldsymbol{e_8}$	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_{12}$	$-e_{13}$	$-e_{14}$	$-e_{15}$	-1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
$\boldsymbol{e_9}$	e_9	e_8	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$	$-e_1$	-1	$-e_3$	e_2	$-e_5$	e_4	e_7	$-e_6$
$\boldsymbol{e_{10}}$	e_{10}	e_{11}	e_8	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}	$-e_2$	e_3	-1	$-e_1$	$-e_6$	$-e_7$	e_4	e_5
$\boldsymbol{e_{11}}$	e_{11}	$-e_{10}$	e_9	e_8	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}	$-e_3$	$-e_2$	e_1	-1	$-e_7$	e_6	$-e_5$	e_4
$\boldsymbol{e_{12}}$	e_{12}	e_{13}	e_{14}	e_{15}	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_4$	e_5	e_6	e_7	-1	$-e_1$	$-e_2$	$-e_3$
$\boldsymbol{e_{13}}$	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	e_8	e_{11}	$-e_{10}$	$-e_5$	$-e_4$	e_7	$-e_6$	e_1	-1	e_3	$-e_2$
$\boldsymbol{e_{14}}$	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	e_8	e_9	$-e_6$	$-e_7$	$-e_4$	e_5	e_2	$-e_3$	-1	e_1
$\boldsymbol{e_{15}}$	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	e_8	$-e_7$	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	-1

Whether there exist other 16-dimensional algebras not generated by the Cayley-Dickson construction was answered in the negative in 1960. If we introduced the following diagram

3_a, 3_b, 1_s
1_t
3_c, 3_d, 1_u

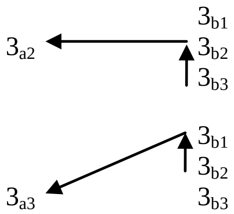
where octonions 3_a, 3_b give $\pm 1_t$ and 3_c, 3_d also gives $\pm 1_t$, allocation 1_s, 1_t 1_u is a quaternion, then we have for octonions 3_a 3_c gives $\pm 1_s$; 3_b, 3_d gives $\pm 1_s$, then 3_c and 3_b is matched with $\pm 1_u$ and 3_a and 3_d with $\pm 1_u$, then this configuration is inconsistent in all configurations.

Irrespective of this result, the Cayley-Dickson construction generating a 64-dimensional algebra shows that this is not a division algebra, since in particular this contains the sedenions as a subalgebra. However, a 64-novanion has $63 = 3^2 \times 7$ non-scalar basis elements, and we will see that novanion algebras of this type are consistent. A 64-novanion is given by the cube with slices

3_a, 3_b, 1_p	$3'_a$, $3'_b$, $1'_p$	$3''_a$, $3''_b$, $1''_p$
3_c, 3_d, 1_q	$3'_c$, $3'_d$, $1'_q$	$3''_c$, $3''_d$, $1''_q$
3_e, 3_f, 1_r	$3'_e$, $3'_f$, $1'_r$	$3''_e$, $3''_f$, $1''_r$.

To evaluate a typical slice algebra, if we take the leftmost array above, we know that 3_b is not a quaternion, so we will build an override structure for the composition of two elements in 3_b. Within this slice 3_b belongs to three octonionic arrangements, those given by 3_a, 3_b, 1_p, or 3_c, 3_b, 1_r, or 3_e, 3_b, 1_q, so we need to select an override on the nonquaternion 3_b, so that when two elements are multiplied within it, just one allocation to an octonionic structure is selected.

We will need to look at this typical example in detail, so denote the three elements of 3_b by 3_{b1}, 3_{b2} and 3_{b3}. We will display the 3 elements of 3_b combining in pairs to form arrows with the following typical structures. We will choose at first, arbitrarily, a link to the 3_a, 3_b, 1_p octonionic structure. Of course, two arrows shown below combine to give an oriented quaternion triple, for which reversal of arrows leads to a minus value.



The central triples 3_d and 3_f have similar structures, mapping to separate values in 3_c and 3_e respectively. We have stated the 1_p, 1_q and 1_r elements combined with 3_b give on composition with one element of 3_b the octonion structures (3_a, 3_b, 1_p), (3_c, 3_b, 1_r) and (3_e, 3_b, 1_q). Because 3_{b1} links to 3_{a1}, we have to ensure that the link 1_p to 3_{b1} does not also link to 3_{a1}, but this can be arranged.

Alternative structures can be considered. For example, if 3_{b3}, 3_{b1} links to 3_{a1} as before, we could also have 3_{b3}, 3_{b2} linking to 3_{c1} and 3_{b2}, 3_{b1} linking to 3_{e1}. □

5.8. Further investigations.

This conclusion is related to questions similar to those about division algebras in homotopy theory, and we will investigate connections with the following circumstances

(i) The homotopy group for a sphere.
(ii) Steenrod square operations.
(iii) Parallelisation of vector fields on spheres.
(iv) Fiber bundle structures.
(v) K(X) as a universal description for objects described by semigroups.
(vi) Bott periodicity.
(vii) Universal Quillen homotopy.

An investigation of issues (i) to (vii) above will be given in *Number, space and logic* [Ad18].

The exceptional Lie algebras G_2, F_4, E_6, E_7 and E_8 are related to the existence of division algebras limited in number to those embedded within the octonions [Wi09], [CSM95].

We have seen in chapter IV that for matrices A, B and C the Lie bracket

$$[AB] = AB - BA$$

satisfies the Jacobi identity

$$[[AB]C] + [[BC]A] + [[CA]B] = 0. \qquad (1)$$

For octonions, we do not have a matrix algebra, but we might wish to form Lie brackets from them satisfying (1). From section 3, for the octonions the only nonquaternionic triple is $e_1e_3e_5$ and

$$[[e_1\ e_3]\ e_5] + [[e_3\ e_5]\ e_1] + [[e_5\ e_1]\ e_3] = 2[e_2\ e_5] + 2[e_4\ e_1] + 2[e_6\ e_3]$$
$$= -12e_7, \qquad (2)$$

so this does not satisfy (1).

In order to create a viable Lie bracket we note that we have constructed the octonions from the quaternions by the Cayley-Dickson construction in section 3. Thus, if we apply an inverse Cayley-Dickson construction to retrieve a pair of quaternions from the octonions, since the quaternions are representable by matrices, on each item of this pair we can create a Lie bracket satisfying (1). The information we might wish to keep in these Lie brackets could also be formed from the sum, difference, matrix product, and the product of a matrix by an inverse matrix of the pair, or any combination of these.

A better solution is to take the Lie brackets of equation (2) (mod 12). □

The 10-novanions we have considered contain the quaternions as a subalgebra. Even when an override structure is imposed, its three elements constitute a quaternion. For the octonion structures we have considered, these also contain a quaternion, and this still applies if a nonstandard override is applied. Thus the n-novanions all contain quaternion subalgebras.

The implications of the existence of algebras of novanion type for the classification of Lie algebras and of simple groups will be addressed in [Ad18].

5.9. Exercises.

(A) This exercise is computationally exhausting on paper. An equation-solver might be more efficient. In equations (6) to (15) of section 4, put $a = p = 0$, $b = b$, $c = d = b' = 1$, $c' = d' = b'' = c'' = d'' = 0$. Including the value of b, what is the solution of equation (5)?